A Note on Ramsey Numbers for Books

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Abstract

The *book with* n *pages* B_n is the graph consisting of n triangles sharing an edge. The *book Ramsey number* $r(B_m, B_n)$ is the smallest integer r such that either $B_m \subset G$ or $B_n \subset \overline{G}$ for every graph G of order r. We prove that there exists a positive constant c such that $r(B_m, B_n) = 2n + 3$ for all $n \geq c\,m$.

1 Introduction

The graph $B_n = K_1 + K_{1,n}$, consisting of n triangles sharing a common edge, is known as the *book* with n *pages*. The book Ramsey number $r(B_m, B_n)$ is the smallest integer r such that either $B_m \subset G$ or $B_n \subset \overline{G}$ for every graph G of order r. The study of Ramsey numbers for books was initiated in [7] and continued in [5]. The following results are known.

Theorem 1 (Rousseau, Sheehan). *For all* $n > 1$, $r(B_1, B_n) = 2n + 3$.

Theorem 2 (Parsons, Rousseau, Sheehan). *If* $2(m + n + 1) > (n - m)^3/3$ *then* $r(B_m, B_n) \leq 2(m + n + 1)$. *More generally,*

$$r(B_m, B_n) \leq m + n + 2 + \left\lfloor \tfrac{2}{3}\sqrt{3(m^2 + mn + n^2)} \right\rfloor . \tag{1}$$

If $4n + 1$ *is a prime power, then* $r(B_n, B_n) = 4n + 2$. *If* $m \equiv 0 \pmod 3$ *then* $r(B_m, B_{m+2}) \leq 4m + 5$.

The more general upper bound (1) was noted by Parsons in [6]. In looking for cases where equality holds in (1) or in other cases covered by Theorem 1, it is natural to consider the class of strongly regular graphs. A (v, k, λ, μ) *strongly regular graph* (SRG) is a graph with v vertices that is regular of degree k in which any two distinct vertices have λ common neighbors if they are adjacent and μ common neighbors if they are nonadjacent. Thus if a (v, k, λ, μ) graph exists then

$$r(B_{\lambda+1}, B_{v-2k+\mu-1}) \geq v + 1.$$

Inspection of a table known strongly regular graphs [4] yields a number of exact values for book Ramsey numbers.

Corollary 1. *In addition those cases where* $4n + 1$ *is a prime power and* $r(B_n, B_n) = 4n + 2$ $(n = 1, 2, 3, 4, 6, \ldots, 69)$*, Theorem 2 gives the following exact values for* $r(B_m, B_n)$ *in which the lower bound comes from a strongly regular graph of order at most* 280.

(m, n)	$r(B_m, B_n)$	(v, k, λ, μ)
(2,5)	16	(15,6,1,3)
(3,5)	17	(16,6,2,2)
(4,6)	22	(21,10,3,6)
(7,10)	36	(35,16,6,8)
(11,11)	46	(45,22,10,11)
(14,17)	64	(63,30,13,15)
(23,26)	100	(99,48,22,24)
(22,37)	120	(119,54,21,27)
(29,38)	136	(135,64,28,32)
(34,37)	144	(143,70,33,35)
(47,50)	196	(195,96,46,48)
(46,58)	210	(209,100,45,50)
(56,56)	226	(225,112,55,56)
(38,82)	244	(243,110,37,60)
(62,65)	256	(255,126,61,63)
(69,71)	281	(280,135,70,60)

The starting point for this paper is Theorem 1 together with the following pair of results from [5].

Theorem 3 (Faudree, Rousseau, Sheehan).

$$r(B_2, B_n) \le \begin{cases} 2n+6, & 2 \le n \le 11, \\ 2n+5, & 12 \le n \le 22, \\ 2n+4, & 23 \le n \le 37, \\ 2n+3, & n \ge 38. \end{cases}$$

Theorem 4 (Faudree, Rousseau, Sheehan). *If $m > 1$ and*

$$n \ge (m-1)(16m^3 + 16m^2 - 24m - 10) + 1,$$

then $r(B_m, B_n) = 2n + 3$.

From these results, we see that for each m there exists a smallest positive integer $f(m)$ such that $r(B_m, B_n) = 2n + 3$ for all $n \ge f(m)$. Moreover $f(1) = 2$ and $f(2) \le 38$. Our main purpose here is to prove the following strengthening of Theorem 4.

Theorem 5. *There exists a positive constant c such that $r(B_m, B_n) = 2n+3$ for all $n \ge cm$.*

2 Proofs

For standard terminology and notation, see [2]. For $v \in V(G)$ we denote the neighborhood of v by $N(v)$ and the degree of v by $\deg(v)$. If needed, we shall use a subscript to identify the graph in question; for example, $N_G(v)$ denotes the neighborhood of v in G. Given two disjoint sets $U, W \subset V(G)$, let $e(U, W) = |\{uw \in E(G) | \ u \in U, \ w \in W\}|$. The subgraph of G induced by $X \subset V(G)$ will be denoted by $G[X]$. Given graphs G and H, let $M_G(H)$ denote the number of induced subgraphs of G that are isomorphic to H. The number of pages in the largest book contained in G will be called the *book size* of G and this will be denoted by $bs(G)$. It is convenient to identify the graph and its complement in terms of edge colorings of a complete graph. In this framework, $r(B_m, B_n)$ is the smallest r such that in every $(R, B) =$ (red, blue) coloring of $E(K_r)$, either $bs(R) \ge m$ or $bs(B) \ge n$. Theorem 5 clearly follows from the following fact.

Theorem 6. *Suppose m and n are positive integers satisfying $n \ge 10^6 m$. If (R, B) is any two-coloring of $E(K_n)$ then either $bs(R) > m$ or $bs(B) \ge n/2 - 2$.*

In view of the case $R = K(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ the conclusion $bs(B) \ge n/2 - 2$ is best possible. The proof of Theorem 6 uses the following counting result.

Lemma 1. *Let G be a graph with p vertices and q edges that satisfies $bs(G) \leq m$. For $0 < \lambda < 1$ suppose that $\delta(G) \geq \lambda p$ so $q \geq \lambda p^2/2$. If $p > 5(2\lambda+1)/\lambda^2$ then*

$$M_G(C_4) > \left(\frac{\lambda^3 p^2}{5} - \frac{m^2}{2}\right) q.$$

Proof. For distinct vertices $u, v \in V(G)$ let $c(u,v) = |N_G(u) \cap N_G(v)|$. Then

$$\sum_{\{u,v\}} \binom{c(u,v)}{2} = 2M_G(C_4) + 6M_G(K_4) + 2M_G(B_2) \quad \text{and}$$

$$\sum_{uv \in E} \binom{c(u,v)}{2} = 6M_G(K_4) + M_G(B_2),$$

from which we get

$$M_G(C_4) = \frac{1}{2} \sum_{\{u,v\}} \binom{c(u,v)}{2} - \sum_{uv \in E} \binom{c(u,v)}{2} + 3M_G(K_4). \tag{2}$$

Since $bs(G) \leq m$,

$$\sum_{uv \in E} \binom{c(u,v)}{2} \leq q \binom{m}{2} < \frac{qm^2}{2}.$$

Note that

$$\sum_{\{u,v\}} c(u,v) = \sum_{v \in V(G)} \binom{\deg_G(v)}{2} \geq p \binom{2q/p}{2} = q(2q/p - 1) \geq q(\lambda p - 1) := x,$$

so by convexity,

$$\sum_{\{u,v\}} \binom{c(u,v)}{2} \geq \binom{p}{2} \binom{x/\binom{p}{2}}{2} = \frac{x}{2} \left(\frac{x}{\binom{p}{2}} - 1 \right).$$

Since

$$\frac{x}{\binom{p}{2}} > \frac{2x}{p^2} = \frac{2q(\lambda p - 1)}{p^2} \geq \lambda(\lambda p - 1),$$

we have

$$\sum_{\{u,v\}} \binom{c(u,v)}{2} > \frac{q(\lambda p - 1)(\lambda(\lambda p - 1) - 1)}{2} > \frac{2\lambda^3 p^2 q}{5}.$$

Note. The last inequality is clear if $\lambda^2 p - 10\lambda - 5 \geq 0$, and hence it holds since we have required $p \geq 5(2\lambda+1)/\lambda^2$. In view of (2) we have

$$M_G(C_4) > \left(\frac{\lambda^3 p^2}{5} - \frac{m^2}{2}\right) q,$$

as claimed. $\square$

Proof of Theorem 6. Suppose $n \geq 10^6 m$ and that (R, B) is a two-coloring of $E(K_n)$ such that $bs(R) \leq m$. We shall prove that $bs(B) \geq n/2 - 2$. Let $\mathcal{H} = C_4 \cup K_1$.

Claim 1. *If $bs(B) \leq n/2 - 2$ then $M_R(\mathcal{H}) \leq 4mM_R(C_4)$.*

Note. The hypothesis $bs(G) \leq n/2 - 2$ rather than, as one might naturally expect, $bs(G) < n/2 - 2$, is made for convenience.

Proof. Suppose $M_R(\mathcal{H}) > 4mM_R(C_4)$. Then there exists an induced $C_4 = (u, v, w, z)$ such that

$$|N_B(u) \cap N_B(v) \cap N_B(w) \cap N_B(z)| \geq 4m + 1.$$

Since $bs(B) \leq n/2 - 2$ we have

$$|N_B(u) \cap N_B(w)| \leq n/2 - 2 \qquad \text{and} \qquad |N_B(v) \cap N_B(z)| \leq n/2 - 2.$$

It then follows that there are at least $4m+1$ vertices outside of $\{u, v, w, z\}$ that are adjacent in R to at least one of u, w and at least one of v, z. This gives $m + 1$ or more red triangles on at least one of the four edges uv, vw, wz, zu, and thus the desired contradiction. □

It is known that for any graph G of order n,

$$M_G(C_4) \leq \binom{\lfloor n/2 \rfloor}{2}\binom{\lceil n/2 \rceil}{2} < \frac{n^4}{64}.$$

See [3] for a proof of the more general result

$$M_G(K_{m,m}) \leq \binom{\lfloor n/2 \rfloor}{m}\binom{\lceil n/2 \rceil}{m}.$$

Hence by Claim 1,

$$M_R(\mathcal{H}) < \frac{mn^4}{16}$$

or else $bs(B) > n/2 - 2$.

Claim 2. *If $bs(B) \leq n/2 - 2$ then R has at most $n/20$ vertices of degree $9n/20$ or less.*

Proof. Let v be any vertex of degree $9n/20$ or less in R and let $X = N_B(v)$. Then $B[X]$ has maximum degree at most $n/2 - 2$ so $G = G(v) = R[X]$ has minimum degree δ satisfying

$$\delta \geq |X| - 1 - \frac{n}{2} + 2 \geq n - 1 - \frac{9n}{20} - 1 - \frac{n}{2} + 2 = \frac{n}{20}$$

and (since $|X|+1 \geq 11n/20$)

$$\delta \geq |X|+1-\frac{n}{2} \geq |X|+1-\frac{1}{2}\left(\frac{20(|X|+1)}{11}\right) = \frac{|X|+1}{11}.$$

Let us check that Lemma 1 applies to G. Take $\lambda = 1/11$ and $p = |X| \geq 11n/20$. Then $p > 5(2\lambda+1)/\lambda^2$ holds provided $n \geq 1302$. This is certainly the case since $n \geq 10^6 m$. Using $m \leq n/10^6$, Lemma 1 gives

$$M_G(C_4) > \left(\frac{1}{5}\cdot\frac{1}{11^3}\left(\frac{11n}{20}\right)^2 - \frac{1}{2}\left(\frac{n}{10^6}\right)^2\right)\frac{1}{2}\left(\frac{11n}{20}\right)\left(\frac{n}{20}\right) \approx \frac{n^4}{640,000}.$$

Suppose more than $n/20$ vertices in R have degree $9n/20$ or less. Then

$$\frac{mn^4}{16} > M_R(\mathcal{H}) = \sum_v M_{G(v)}(C_4) > \sum_{\deg(v)\leq 9n/20} M_{G(v)}(C_4) > \frac{n}{20}\cdot\frac{n^4}{640,000},$$

so $n < 8\cdot 10^5 m$, a contradiction. $\square$

Let $S = \{v|\ \deg_R(v) > 9n/20\}$. From Lemma 3 we know that $|S| > 19n/20$, so the minimum degree of $R[S]$ satisfies

$$\delta \geq \frac{9n}{20} - (n-|S|) > \frac{2n}{5} \geq \frac{2|S|}{5}.$$

Now we use the following result of Andrásfai, Erdős and Sós [1].

Theorem 7 (Andrásfai, Erdős, Sós). *Suppose $r \geq 3$. For any graph G of order n, at most two of the following properties can hold:*

$$\text{(i)}\ K_r \not\subseteq G, \qquad \text{(ii)}\ \delta(G) > \frac{3r-7}{3r-4}n, \qquad \text{(iii)}\ \chi(G) \geq r.$$

Note. In particular, a triangle-free graph G with $\delta(G) > 2|V(G)|/5$ is bipartite.

Now we are prepared to complete the proof of Theorem 6. It is easy to see that $R[S]$ has no triangle. If $T = \{u,v,w\}$ is a triangle in $R[S]$ and U is the set of $n-3$ vertices outside T, then

$$3(9n/20-2) < e_R(T,U) \leq 3(m-1)\cdot 2 + (n-3(m-1)) = n+3(m-1),$$

or $7n/20 < 3m+3$, which is false. Hence $R[S]$ is bipartite by Theorem 7. Let S_1 and S_2 denote the two color classes of $R[S]$. Put $v \in T_1$ if v is adjacent in B to every vertex of S_1.

Then for the remaining vertices put $v \in T_2$ if v is adjacent in B to every vertex of S_2. Let $W_1 = S_1 \cup T_1, W_2 = S_2 \cup T_2$, and let X denote the set of vertices in neither W_1 nor W_2. If $X = \varnothing$ then we may assume that $|W_1| \geq n/2$. In this case it is clear that $bs(B) \geq n/2 - 2$.

We are left to consider the case $X \neq \varnothing$. For $u \in S$ let $Z(u) = N_B(u) \cap X$. For distinct vertices $u, v \in S_1$, consideration of the blue book on uv shows that

$$\begin{aligned} bs(B) &\geq |S_1| - 2 + |T_1| + |Z(u) \cap Z(v)| \\ &\geq |S_1| - 2 + |T_1| + |Z(u)| + |Z(v)| - |X|. \end{aligned}$$

Summing over all pairs $u, v \in S_1$ and computing the average, we find

$$\begin{aligned} bs(B) &\geq |S_1| - 2 + |T_1| + \frac{2(|S_1||X| - e_R(S_1, X))}{|S_1|} - |X| \\ &= |S_1| + |T_1| + |X| - 2 - \frac{2e_R(S_1, X)}{|S_1|}. \end{aligned}$$

Similarly,

$$bs(B) \geq |S_2| + |T_2| + |X| - 2 - \frac{2e_R(S_2, X)}{|S_2|}.$$

Note that $|S_1| < n/2$ or else we are done at the outset; similarly $|S_2| < n/2$. Hence

$$|S_1| = |S| - |S_2| > \frac{19n}{20} - \frac{n}{2} = \frac{9n}{20},$$

and likewise $|S_2| > 9n/20$. Consequently

$$\begin{aligned} bs(B) &> |S_1| + |T_1| + |X| - 2 - \frac{40e_R(S_1, X)}{9n}, \\ bs(B) &> |S_2| + |T_2| + |X| - 2 - \frac{40e_R(S_2, X)}{9n}. \end{aligned}$$

Addition then gives

$$2\, bs(B) > n - 4 + |X| - \frac{40e_R(S, X)}{9n}.$$

Hence $e_R(S, X) > 9n|X|/40$ or else the proof is complete.

Thus we assume $e_R(S, X) > 9n|X|/40$ and now seek a companion bound on $e_R(S, X)$. For each $x \in X$ there is at least one $v \in S_1$ such that $xv \in R$. Since $|N_R(v) \cap S_2| \geq 2|S|/5$,

consideration of the red book on xv shows that

$$\begin{aligned} bs(R) &\geq |N_R(x) \cap N_R(v) \cap S_2| \\ &= |N_R(x) \cap S_2| + |N_R(v) \cap S_2| - |S_2| \\ &\geq |N_R(x) \cap S_2| + \frac{2|S|}{5} - |S_2|. \end{aligned}$$

Taking the average over $x \in X$, we obtain

$$bs(R) \geq \frac{e_R(S_2, X)}{|X|} + \frac{2|S|}{5} - |S_2|.$$

In exactly the same way,

$$bs(R) \geq \frac{e_R(S_1, X)}{|X|} + \frac{2|S|}{5} - |S_1|.$$

Hence

$$2m \geq 2bs(R) \geq \frac{e_R(S, X)}{|X|} - \frac{|S|}{5}.$$

Thus

$$e_R(S, X) \geq 2m|X| + \frac{|S||X|}{5}.$$

From the two bounds for $e_R(S, X)$, we obtain

$$\frac{9n|X|}{40} < e_R(S, X) \leq \frac{|S||X|}{5} + 2m|X| < \frac{n|X|}{5} + 2m|X|.$$

By assumption $|X| > 0$, so

$$\frac{9n}{40} < \frac{n}{5} + 2m,$$

which is false. $\square$

3 Concluding Remarks

The determination of the best constant c in Theorem 5 is open, as are other basic problems on book Ramsey numbers stated in [5]. In particular, it is unknown whether or not there exists a constant C such that $r(B_m, B_n) \leq 2(m + n) + C$ for all m, n.

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